ESTIMATION OF BOUNDS FOR THE GEOPOTENTIAL COEFFICIENTS

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ABSTRACT

Cholshevnikov[1,2] has published the results of analytic studies demonstrating that bounds on the zonal geopotential coefficients decrease at least as rapidly as $1/n^3$ and bounds on the tesseral coefficients at least as rapidly as $1/mn^2$. We show that these bounds are almost certainly conservative, and then proceed to develop a number of possible modifications designed to incorporate extensive data on the Earth's density distribution into the analysis. None of the modifications are implemented; the most promising is, however, discussed at some length.

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I. INTRODUCTION

The normalized geopotential coefficients can be given in terms of the density of the Earth, $\rho(r, z, \varphi)$, and the normalized associated Legendre functions, $P_{nm}(z)$, used by geodesists, as

$$\left\{ \begin{matrix} C_{nm} \\ S_{nm} \end{matrix} \right\} = \frac{1}{2n+1} \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{A(z,\varphi)} \left[\rho(r, z, \varphi) \frac{r^{n+2}}{A_e^n} P_{nm}(z) \left\{ \begin{matrix} \cos m\varphi \\ \sin m\varphi \end{matrix} \right\} \right] d\varphi \, dz \, dr \tag{1.1}$$

where

A_e	=	radius of the smallest sphere centered at the origin and circumscribing the Earth
z	=	$\cos\theta$; θ = polar angle of spherical coordinates
φ	=	longitude
$A(z, \varphi)$	=	distance of that point on the surface of the Earth, defined by z and φ, from the origin

Knowledge of the behavior of these coefficients as functions of n and m would be useful in many ways, particularly in providing a rational basis for truncation of the geopotential in applications to orbital mechanics, and in the formulation of procedures for geopotential modeling.

Since the density, ρ, is an empirical quantity, an analytic evaluation of the coefficients from Eq. (1.1) is not possible. With sufficiently detailed density data, numerical integration of the right hand side of Eq. (1.1) could be used to construct a table of values for the coefficients. Although sufficient data for such a calculation are not presently available, a considerable amount of data does exist, on the basis of which various density models for the Earth have been constructed. The purpose of this report is to discuss how the available data and models might be used to investigate the behavior, not of the coefficients themselves, but of bounds on the coefficients. It has been generally supposed for a long time that the geo-

potential coefficients decrease as n increases, a conjecture supported by empirical estimates based on examination of tables of empirically determined coefficients. The most sophisticated analytic estimates, to my knowledge, are those of Cholshevnikov,[1,2] who gives four mathematically rigorous derivations starting from Eq. (1.1). His results are on the rate of decrease of upper bounds on the magnitudes of the coefficients, and we shall describe them in some detail in the next section. Here we wish only to comment briefly on the approach, and introduce the mathematical background necessary for his development.

A bound of, say, A/n^2 for $|C_{nm}|$ is not the same as a decrease of $|C_{nm}|$ as A/n^2; $|C_{nm}|$ might decrease more rapidly; or, perhaps, starting from a value much lower than the bound, it might increase for awhile; or it could even have a damped oscillatory character. However, whatever the variation of $|C_{nm}|$ with n, $|C_{nm}|$ would be constrained to lie below A/n^2, and the bound would imply $\lim_{n \to z} C_{nm} = 0$.

Cholshevnikov's bounds are obtained by using standard inequality relationships on the absolute value of an integral, bounds on $P_{nm}(z)$ and/or its integral from -1 to $+1$, and, for the sharper results, a mean value theorem for integrals based on the concept of a monotone function. A monotonically increasing function of x satisfies the inequality

$$f(x_1) \geq f(x_2) \quad \text{for} \quad x_1 > x_2 \tag{1.2}$$

It differs from a strictly increasing function since there may exist one or more intervals over which it remains constant. A monotonically decreasing function is defined by reversing the second inequality. We now state the mean value theorem; a proof will be found in Reference 3.

Theorem: If $F(x)$ and $G(x)$ are continuous in the interval (a, b), and also if $F(x)$ is monotone (either increasing or decreasing) in this interval, then there exists an x_1 in the interval such that

$$\int_a^b F(x)\, G(x)\, dx = F(a) \int_a^{x_1} G(x)\, dx + F(b) \int_{x_1}^b G(x)\, dx \tag{1.3}$$

Clearly, the bounds on C_{nm} and S_{nm} must involve the density function and the shape of the Earth defined by $A(z, \varphi)$, another empirical quantity. Cholshevnikov eliminates $A(z, \varphi)$ by using A_e as the upper bound for the integration with respect to r in Eq. (1.1). This implies that the density function must be set to zero in the gap between the surface of the Earth and its circumscribing sphere. Cholshevnikov incorporates the density function in his analysis through the "global" functions of density

$$\rho_{max}, \text{ a global constant } \quad (\rho_{min} = 0)$$

$$v_\varphi(r, \theta, \varphi) = \int_0^\varphi \left| \frac{\partial \rho}{\partial \varphi} \right| d\varphi = \text{variation}^* \text{ of } \rho \text{ with respect to } \varphi \tag{1.4}$$

$$v_z(r, z. \varphi) = \int_0^z \left| \frac{\partial \rho}{\partial z} \right| dz = \text{variation of } \rho \text{ with respect to } z$$

Clearly v_φ and v_z are monotonically increasing functions of φ and z, respectively, and a little thought will show that $(v_\varphi - \rho)$ and $(v_z - \rho)$ are also monotonically increasing with respect to φ and z, respectively. These comments perhaps already suggest how Cholshevnikov derives his bounds. Since several modifications of his analysis are almost immediately apparent, we shall devote Section II to an abbreviated derivation of his results which, while rigorous, have two defects:

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* For a more rigorous definition of "variation," see Reference 4.

1. One would conjecture that they must be fairly conservative since the fact that the integrand in Eq. (1.1) has one sign over approximately half of the region of integration, and the opposite sign over the rest, appears to have been fully exploited only in his Theorem II.

2. The parameter, ρ_{max}, can be reasonably well estimated. Adequate data for the estimation of bounds on $v_{\varphi}(r, \theta, 2\pi)$ and $v_z(r, 1, \varphi)$, used in the sharper results, are probably not available.

In Section III we discuss some possible modifications of Cholshevnikov's methods, aimed at eliminating the defects mentioned above. The primary change is to introduce local formulations rather than the global formulations implemented by Cholshevnikov. In Section IV, we select what appears to be the most promising of the local formulations for a more detailed analysis. It is beyond the scope of this report to implement any of the suggested formulations. We do indicate, however, how data on the density function and the shape of the Earth might be used in an implementation. In the last section, we make a few comments on "worst case" constructions and their rôle in deriving bounds on the geopotential coefficients.

II. CHOLSHEVNIKOV'S BOUNDS FOR GEOPOTENTIAL COEFFICIENTS

The first of Cholshevnikov's papers[1] gives two bounds for the coefficients of the zonal harmonics, one in terms of ρ_{max} and the other (which is sharper) in terms of the variation v_z. The second paper carries through the corresponding analysis for the coefficients of the tesseral harmonics, the major difference in the two investigations being that the variation v_φ is used in the second. We shall state these four theorems and sketch the proofs, modified to apply to normalized coefficients (Cholshevnikov used conventional associated Legendre functions and unnormalized coefficients). The analysis applies to the defining equation (1.1), for the geopotential coefficients, written in the form

$$\begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = \frac{1}{2n+1} \int_0^{A_e} \frac{r^{n+2}}{A_e^n} \left\{ \int_{-1}^{1} \int_0^{2\pi} \rho(r, z, \varphi)\, P_{nm}(z) \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} d\varphi\, dz \right\} dr \quad (2.1)$$

This form differs from (1.1) in the upper limit of the r-integral. There is no loss of generality; the replacement of $A(z, \varphi)$ by A_e can be compensated by setting the density ρ to zero in the gap between the actual surface of the Earth, defined by

$$r = A(z, \varphi) \quad (2.2)$$

and the circumscribing sphere of radius A_e, since from the definition of A_e

$$A(z, \varphi) \le A_e \quad (2.3)$$

Theorem I: $$|C_{n0}| \le \frac{4\pi \rho_{max} A_e^3}{(2n+1)(n+3)} \quad (2.4)$$

Proof: From Eq. (2.1), $S_{n0} = 0$ and

$$C_{n0} \leq \frac{1}{2n+1} \int_0^{A_e} \frac{r^{n+2}}{A_e^n} \left\{ \int_{-1}^{1} \int_0^{2\pi} |\rho(r, z, \varphi)| \cdot |P_{nm}(z)| \, d\varphi \, dz \right\} dr$$

$$< \frac{2\pi \rho_{max}}{2n+1} \left[\frac{r^{n+3}}{(n+3)A_e^n} \right]_0^{A_e} \int_{-1}^{1} |P_n(z)| \, dz \qquad (2.5)$$

from which the result follows immediately on use of the inequality

$$\int_0^1 |P_{n0}(z)| \, dz \leq 2 \qquad (2.6)$$

which Cholshevnikov credits to Hobson. Note that the second inequality in (2.5) is strictly "less than" because if $\rho = \rho_{max}$ for all r, z, φ, Eq. (2.1) implies $C_{n0} = 0$ for all n except for $n = 0$.

Theorem II: $$|C_{n0}| \leq \frac{4\pi A_e^3 \sqrt[4]{22}}{(2n+1)(n+3)} \cdot \frac{2 v_{z_{max}} + \rho_{max}}{\sqrt{\pi (n-1)(2n+1)}} \qquad (2.7)$$

where

$$v_{z_{max}} = \mathrm{LUB}(v_z(r, 1, \varphi) \text{ for all } r, \varphi) \qquad (2.8)$$

Note: This theorem implies that $\mathrm{LUB}|C_{n0}|$ decreases as const/n^3 for large n. Cholshevnikov shows, in his first paper, that this theorem is a "best" result in the sense that a density distribution can be constructed for which $|C_{n0}|$ itself decreases as $1/n^3$. We comment in more detail on this in Section V.

Proof: Rewrite Eq. (2.1) in the form

$$C_{n0} = \frac{1}{2n+1} \int_0^{A_e} \frac{r^{n+2}}{A_e^n} \int_0^{2\pi} \left\{ \int_{-1}^{1} v_z(r, z, \varphi) P_{n0}(z)\, dz \right.$$

$$\left. - \int_{-1}^{1} [v_z(r, z, \varphi) - \rho(r, z, \varphi)] P_{n0}(z)\, dz \right\} d\varphi\, dr \tag{2.9}$$

Since both v_z and $v_z - \rho$ are monotonically increasing functions of z, the expression in "curly" brackets may be replaced by

$$v_z(r, 1, \varphi) \int_{z_1}^{1} P_{n0}(z)\, dz - [-\rho(r, -1, \varphi)] \int_{-1}^{z_2} P_{n0}(z)\, dz$$

$$- [v_z(r, 1, \varphi) - \rho(r, 1, \varphi)] \int_{z_2}^{1} P_n(z)\, dz \tag{2.10}$$

where use has been made of the mean value theorem, stated in the Introduction, and of the fact that

$$v_z(r, -1, \varphi) = 0 \tag{2.11}$$

Further, since

$$\int_{-1}^{1} P_n(z)\, dz = 0 = \int_{-1}^{z_2} P_n(z)\, dz + \int_{z_2}^{1} P_n(z)\, dz \tag{2.12}$$

we can combine the last two integrals of Eq. (2.10) to obtain

$$v_z(r, 1, \varphi) \left[\int_{z_1}^{1} P_n(z)\, dz - \int_{z_2}^{1} P_n(z)\, dz \right]$$

$$+ [\rho(r, 1, \varphi) - \rho(r, -1, \varphi)] \int_{z_2}^{1} P_n(z)\, dz \tag{2.13}$$

for the expression in curly brackets. Cholshevnikov now makes use of another inequality

$$\left|\int_a^1 P_n(z)\, dz\right| \le \frac{2\sqrt[4]{22}}{\sqrt{\pi(n-1)(2n+1)}} \tag{2.14}$$

to obtain

$$|C_{n0}| \le \frac{1}{2n+1}\int_0^{2\pi}[2v_{z_{max}} + \rho_{max}]\frac{2\sqrt[4]{22}}{\sqrt{\pi(n-1)(2n+1)}}\, d\varphi\, dr \tag{2.15}$$

from which the desired result follows immediately. The reader is referred to Cholshevnikov's paper for the derivation of the inequality (2.14). Note that we have dropped the subscript 0 on P_{n0}, since for $m = 0$, the associated functions become Legendre polynomials.

Theorem III: $$\left\{\begin{matrix}|C_{nm}|\\ |S_{nm}|\end{matrix}\right\} < \frac{4\pi\sqrt{2}\,\rho_{max} A_e^3}{(2n+1)(m+3)} \quad \text{for } m>0 \tag{2.16}$$

Proof: From Eq. (2.1)

$$C_{nm} + i\, S_{nm} = \frac{1}{2n+1}\int_0^{A_e}\frac{r^{n+2}}{A_e^n}\int_{-1}^1 P_{nm}(z)\left[\int_0^{2\pi} e^{im\varphi}\rho(r, z, \varphi)\, d\varphi\right] dz\, dr \tag{2.17}$$

and hence

$$\left\{\begin{matrix}|C_{nm}|\\ |S_{nm}|\end{matrix}\right\} \le |C_{nm} + i\, S_{nm}| \le \frac{1}{2n+1}\int_0^{A_e}\frac{r^{n+2}}{A_e^n}\int_{-1}^1 P_{nm}(z)\left[\int_0^{2\pi}|\rho(r, z, \varphi)|\, d\varphi\right] dz\, dr$$

$$< \frac{2\pi\rho_{max}}{2n+1}\left[\frac{r^{n+3}}{(n+3)A_e^n}\right]_0^{A_e} 2\sqrt{2} \tag{2.18}$$

where use has been made of another inequality credited to Hobson:

$$\int_{-1}^1 |P_{nm}(z)|\, dz \le 2\sqrt{2} \tag{2.19}$$

Theorem IV: $$\left\{ \begin{array}{l} |C_{nm}| \\ |S_{nm}| \end{array} \right\} < \frac{4\sqrt{2}\, v_{\varphi_{max}} A_e^3}{m(n+3)(2n+1)} \tag{2.20}$$

Proof:

$$\left\{ \begin{array}{l} |C_{nm}| \\ |S_{nm}| \end{array} \right\} \leq \frac{1}{2n+1} \int_0^{A_e} \frac{r^{n+2}}{A_e^n} \int_{-1}^{1} |P_{nm}(z)| \left| \int_0^{2\pi} \rho(r, z, \varphi) \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} d\varphi \right| dz\, dr \tag{2.21}$$

This time, using the variation with respect to φ, we obtain a bound on the φ integral:

$$\left| \int_0^{2\pi} \rho \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} d\varphi \right| = \left| v_{\varphi}(r, z, 2\pi) \int_{\varphi_1}^{2\pi} \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} d\varphi \right.$$

$$\left. + \rho(r, z, 0) \int_0^{z_2} \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} d\varphi - [v(r, z, 2\pi) - \rho(r, z, 2\pi)] \int_{\varphi_2}^{2\pi} \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} d\varphi \right| \tag{2.22}$$

where we have used $v_{\varphi}(r, z, 0) = 0$. From continuity, we have

$$\rho(r, z, 0) = \rho(r, z, 2\pi) \tag{2.23}$$

and hence

$$\left| \int_0^{2\pi} \rho \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} d\varphi \right| = \left| v_{\varphi}(r, z, 2\pi) \int_{\varphi_1}^{\varphi_2} \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} d\varphi + \rho(r, z, 0) \int_0^{2\pi} \left\{ \begin{array}{l} \cos m\varphi \\ \sin m\varphi \end{array} \right\} d\varphi \right|$$

$$= \left| \frac{v_{\varphi}(r, z, 2\pi)}{m} \left\{ \begin{array}{l} \sin m\varphi_2 - \sin m\varphi_1 \\ \cos m\varphi_1 - \cos m\varphi_2 \end{array} \right\} \right| \leq \frac{2 v_{\varphi_{max}}}{m} . \tag{2.24}$$

Proceeding now as in Theorem III, the desired result is easily obtained.

We now combine the methods used by Cholshevnikov for Theorems II and III to obtain a fifth theorem:

Theorem V:

$$\left\{ \begin{matrix} |C_{nm}| \\ |S_{nm}| \end{matrix} \right\} \leq \frac{2\pi A_e^3}{(2n+1)(n+3)} [2 v_{z_{max}} + \rho_{max}] \left| \int_{-1}^{1} P_{nm}(z)\, dz \right| \tag{2.25}$$

Proof:

$$|C_{nm} + i\, S_{nm}| \leq \frac{1}{2n+1} \int_0^{A_e} \frac{r^{n+2}}{A_e^n} \int_0^{2\pi} |e^{i m \varphi}| \cdot \left| \int_{-1}^{1} \rho(r, z, \varphi)\, P_{nm}(z)\, dz \right| d\varphi\, dr \tag{2.26}$$

or

$$\left\{ \begin{matrix} |C_{nm}| \\ |S_{nm}| \end{matrix} \right\} \leq \frac{1}{2n+1} \int_0^{A_e} \frac{r^{n+2}}{A_e^n} \int_0^{2\pi} [2 v_{z_{max}} + \rho_{max}] \cdot \left| \int_{-1}^{1} P_{nm}(z)\, dz \right| d\varphi\, dr$$

where the z-integral has been treated as in Theorem II. Note that use of inequality (2.20) would yield a result inferior to Theorem IV. Presumably, Cholshevnikov would have included this theorem also had an estimate of $\left|\int_{-1}^{1} P_{nm}(z)\, dz\right|$ decreasing faster than $1/m$ been available to him. Analytic bounds on $\int_{-1}^{1} P_{nm}(z)\, dz$ are not known to me.

Of his various theorems, Cholshevnikov proved only that Theorem II is a best result, i.e., that density functions (however "unrealistic") exist for which the equality, rather than the inequality, holds. If Theorem II is "best," Theorem I cannot be "best." In fact, since both Theorems I and III involve only ρ_{max} and bounds on $\int |P_{nm}(z)|\, dz$ (which do not exploit the oscillatory character of P_{nm}), both can certainly be sharpened. Theorems IV and V make use of a variation of ρ and include the oscillatory character of both $P_{nm}(z)$ and $\cos(m\varphi)$ or $\sin(m\varphi)$; it is possible that at least one of them is a "best" result. The variations $v_{z_{max}}$ and $v_{\varphi_{max}}$ must, however, be enormous numbers, and even rough estimates of their values are probably impossible to make from available density data. In

spite of this, Theorem II, involving $v_{z_{max}}$ is a "best" result, at least for the order of magnitude of the rate of decrease. We shall return to this point in Section V.

The theorems discussed in this section all involve constant factors depending upon "global" functions of the density: ρ_{max}, $v_{z_{max}}$, and $v_{\varphi_{max}}$. In the following section, we propose a method for sharpening at least the constant factors by constructing a local rather than a global formulation, with the possibility of using local rather than global bounds on the density and its variations. Such a formulation should enable us to make more effective use of the available density data.

III. LOCAL PROCEDURES FOR BOUNDING THE GEOPOTENTIAL COEFFICIENTS

Our local formulations will be based on rewriting the defining equation (1.1) for the geopotential coefficients in the equivalent form, for $m > 0$:

$$\left\{ \begin{matrix} C_{nm} \\ S_{nm} \end{matrix} \right\} = \frac{1}{2n+1} \sum_{i=1}^{n-m+1} \sum_{j=1}^{2m-1} I_{ij} \tag{3.1}$$

with

$$I_{ij} = \int_{z_i}^{z_{i+1}} \int_{\varphi_j}^{\varphi_{j+1}} P_{nm}(z) \left\{ \begin{matrix} \cos m\varphi \\ \sin m\varphi \end{matrix} \right\} R_n(z, \varphi)\, d\varphi\, dz \tag{3.2}$$

where

$$R_n(z, \varphi) = \int_0^{A(z,\varphi)} \rho(r, z, \varphi) \frac{r^{n+2}}{A_e^n}\, dr \quad ,$$

the z_i are the $n-m+2$ zeroes of $P_{nm}(z)$ with $z_1 = -1$, $z_{n-m+2} = +1$, (3.3)

the φ_j are the $2m$ zeroes of $\left\{ \begin{matrix} \cos m\varphi \\ \sin m\varphi \end{matrix} \right\}$ in the interval $0 \le \varphi < 2\pi$.

The case $m = 0$ requires special treatment and, since it may not be amenable to a local formulation, it will not be discussed in this report.

The z_i and φ_j define a grid of spherical quadrilaterals (triangles at the poles) bounded by the latitude and longitude lines corresponding to the zeroes of that spherical harmonic which defines C_{nm} (or S_{nm}). Note that this representation is unique for <u>each</u> of the coefficients C_{nm}, S_{nm} as n and m run from zero to whatever upper limit is imposed.

The particular representation of Eq. (1.1) as a double sum has the property that for

$$\begin{aligned} &z_i \le z \le z_{i+1} \qquad P_{nm} \text{ does not change sign} \\ &\varphi_j \le \varphi \le \varphi_{j+1} \qquad \left\{ \begin{matrix} \cos m\varphi \\ \sin m\varphi \end{matrix} \right\} \text{ does not change sign} \end{aligned} \tag{3.4}$$

which implies that the inequality usually present in relating the absolute value of an integral to the integral of its absolute value degenerates for I_{ij} to

$$|I_{ij}| = \pm \int_{z_i}^{z_{i+1}} \int_{\varphi_j}^{\varphi_{j+1}} |P_{nm}(z)| \left\{ \begin{matrix} |\cos m\varphi| \\ |\sin m\varphi| \end{matrix} \right\} R_n(z, \varphi)\, d\varphi\, dz \tag{3.5}$$

Further, the fact that the z_i and φ_j are zeroes of $P_{nm}(z)$ and $\left\{ \begin{matrix} \cos m\varphi \\ \sin m\varphi \end{matrix} \right\}$, respectively, implies that

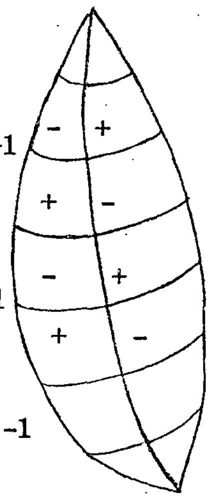
$$I_{i\pm1,j} \text{ and } I_{i,j\pm1} \text{ have signs opposite to that of } I_{ij} \tag{3.6}$$

since one factor of the integrand changes sign if either i or j (but not both) change by 1, while the other factor does not change sign. Similarly,

$$I_{i\pm1,j\pm1} \text{ has the same sign as } I_{ij} \tag{3.7}$$

for all four combinations of the +'s and −'s.

Now let us examine the contribution to the sum over i and j in Eq. (3.1) of two neighboring sectors with common boundary φ_j. Referring to the sketch, we see that the positive contributions come from one zigzag path, and the negative contributions from another. Note that if R_n were constant, the total contribution of this pair of sectors would vanish and, hence, C_{nm} and S_{nm} would also vanish



since there are 2m sectors and, therefore, an integral number m of pairs. Next let us set

$$R_{ij,\max} = \mathrm{LUB}\ R_n(z,\varphi)$$
$$R_{ij,\min} = \mathrm{GLB}\ R_n(z,\varphi) \qquad (3.8)$$

for "block i, j", defined by the inequalities of (3.4). If we take I_{ij} to be negative, as in the sketch,

$$I_{i,j-1} + I_{i,j} < (R_{ij,\max} - R_{ij,\min}) \left[\int_{z_i}^{z_{i+1}} P_{nm}(z)\, dz\right]\left[\int_{\varphi_{j-1}}^{\varphi_j} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} d\varphi\right]$$
$$= \frac{2[R_{ij,\max} - R_{ij,\min}]}{m} \left| \int_{z_i}^{z_{i+1}} P_{nm}(z)\, dz \right| \qquad (3.9)$$

and the more nearly equal are $R_{ij,\max}$ and $R_{ij,\min}$, the more fully can the cancellation effect from integrands of opposite signs come into play. Also, the more accurately the interval integrals between successive zeroes of P_{nm} can be estimated, the more accurately can the n, m dependence of the bounds of the geopotential coefficients be determined. References 5 and 6 discuss various properties of $P_{nm}(z)$ pertinent to this latter problem. Our primary concern here is with the incorporation of data into the analysis — that is, with possible procedures for the estimation of the bounds $R_{ij,\max}$ and $R_{ij,\min}$ of $R_n(z,\varphi)$ for the i, j^{th} block.

We first recall that R_n depends upon the two empirical quantities; $A(z,\varphi)$, which defines the shape of the Earth, and the density function $\rho(r,z,\varphi)$; and

that, in fact, these empirical quantities appear explicitly only in R_n in this formulation. Basic inequalities involving $A(z, \varphi)$ which might be useful are

$$A_{min} < A(z, \varphi) \le B(z) \le A_e \quad (3.10)$$

where $B(z)$ is the distance from the origin to a point with polar angle $\arccos z$ on the smallest oblate ellipsoid with semi-major axis A_e, centered at the origin, and circumscribing the Earth. A_{min} is the radius of the largest sphere, centered at the origin, and contained by the surface of the Earth. The inequalities (3.10) are "global." Local LUB's and GLB's of $A(z, \varphi)$ would clearly be useful, and can probably be fairly well sized over most of the globe. Probably the real key to the estimation of realistic bounds for R_n lies in finding an effective way to handle ρ. A discussion of this problem is given in the next section for the "radial" formulation developed above.

Before discussing the radial formulation in depth, however, we wish to point out that the local approach, including the symmetry properties of the "block" integrals incorporated in Eqs. (3.5), (3.6), and (3.7), can also be given the following formulations:

$$\begin{aligned} I_{ij} &= \int_{z_i}^{z_{i+1}} \int_0^{B(z)} P_{nm}(z) \frac{r^{n+2}}{A_e^n} \Phi_j(r, z)\, dr\, dz \\ &= \int_0^{A_e} \frac{r^{n+2}}{A_e^n} \int_{\varphi_j}^{\varphi_{j+1}} \left\{ \begin{matrix} \cos m\varphi \\ \sin m\varphi \end{matrix} \right\} Z_i(r, \varphi)\, d\varphi\, dr \end{aligned} \quad (3.11)$$

with

$$\Phi_j(r, z) = \int_{\varphi_j}^{\varphi_{j+1}} \rho(r, z, \varphi) \left\{ \begin{matrix} \cos m\varphi \\ \sin m\varphi \end{matrix} \right\} d\varphi \quad (3.12a)$$

$$Z_i(r,\varphi) = \int_{z_i}^{z_{i+1}} \rho(r,z,\varphi)\, P_{nm}(z)\, dz \tag{3.12b}$$

The inequalities (3.10) have been used in writing Eq. (3.11) and, in order to maintain Eq. (1.1), $\rho(r,z,\varphi)$ must be set equal to zero in the gap between the surface of the Earth defined by $A(z,\varphi)$ and the surface defined by $B(z)$ in the first of Eqs. (3.11), and the circumscribing sphere of radius A_e in the second. These formulations lend themselves to the application of the methods used by Cholshevnikov in a local rather than a global manner; that is, ρ_{max} and the variations v_φ and v_z can be defined for the individual blocks to obtain bounds on Φ_j and Z_i.

Proceeding just as in the analysis leading to inequality (3.9) and using the symmetry properties of the blocks, we find the corresponding inequalities

$$I_{i,j-1} + I_{ij} < \left[\Phi_{ij,max} - \Phi_{ij,min}\right] \int_0^{B(z)} \frac{r^{n+2}}{A_e^n}\, dr \cdot \left| \int_{z_i}^{z_{i+1}} P_{nm}(z)\, dz \right|$$

$$< \left[\Phi_{ij,max} - \Phi_{ij,min}\right] \frac{A_e^3}{n+3} \int_{z_i}^{z_{i+1}} \left| P_{nm}(z) \right| dz \tag{3.13}$$

and

$$I_{i,j-1} + I_{ij} < \left[Z_{ij,max} - Z_{ij,min}\right] \int_0^{A_e} \frac{r^{n+2}}{A_e^n}\, dr \cdot \left| \int_{\varphi_j}^{\varphi_{j+1}} \left\{ \begin{matrix} \cos m\varphi \\ \sin m\varphi \end{matrix} \right\} d\varphi \right|$$

$$= \left[Z_{ij,max} - Z_{ij,min}\right] \frac{2A_e^3}{m(n+3)} \tag{3.14}$$

The task of estimating bounds for Z_i and Φ_j appears to be somewhat less tractable than that of estimating bounds for R_n. Further, in the first of these formulations, (3.13), the empirical function $B(z)$ is present in the first of the inequalities

although, as indicated in the second inequality, it can be eliminated at the expense of weakening the bound.

No effort has been made to implement these two formulations; they are suggested because the utilization of the variations v_φ and v_z by Cholshevnikov appeared rather effective, although the difficulty of estimation of global bounds for v_φ and v_z renders his results of theoretical rather than practical interest. The estimation of local bounds, particularly for $n \gg m \gg 1$ (so that the blocks are all small), may be feasible.

IV. INCORPORATION OF DENSITY DATA INTO A LOCAL RADIAL FORMULATION

We designate our principal local formulation, based on R_n, as a local radial formulation for rather obvious reasons: it is local and R_n is the radial part of the defining integral (1.1) for the geopotential coefficients. The function R_n contains all the empirical quantities and, once we can establish bounds on R_n for each block i, j, the bounds on the geopotential coefficients are constructed from a weighted sum of the interval integrals

$$\int_{z_i}^{z_{i+1}} P_{nm}(z)\, dz \tag{4.1}$$

of $P_{nm}(z)$ between successive pairs of its zeroes, as outlined in the previous section. A computer program[6] has been written to calculate the interval integrals and can operate with accuracy adequate for this purpose up to degree and order 40, 40. Analytic estimates of the interval integrals as functions of n and m would be highly desirable. No such estimates are known to me, and their derivation appears to be a problem of considerable difficulty.

The local formulation (3.1) of the defining equations for the geopotential coefficients has somewhat the character of a numerical integration. The original triple integral of Eq. (1.1) has been replaced by a double sum of block integrals:

$$\begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = \frac{1}{2n+1} \sum_{i=1}^{n-m+1} \sum_{j=1}^{2m-1} \int_{z_i}^{z_{i+1}} \int_{\varphi_j}^{\varphi_{j+1}} P_{nm}(z) \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} R_n(z, \varphi)\, d\varphi\, dz \tag{4.2}$$

with

$$R_n(z, \varphi) = \int_0^{A(z,\varphi)} \rho(r, z, \varphi) \frac{r^{n+2}}{A_e^n}\, dr \tag{4.3}$$

No approximations have been introduced up to this point. Notice that the sizes of the blocks are determined by the values assigned to n and m, and may be too large for a conventional numerical integration. This could, of course, be remedied by further subdividing each block. The available density data and data on the shape of the Earth's surface defined by $A(z, \varphi)$ are, however, not adequate to carry out a direct numerical integration of Eqs. (4.2) and (4.3) with sufficient accuracy to give realistic estimates of the values of C_{nm} and S_{nm}. We seek, instead, bounds on R_n to be used as input for calculation of bounds on the coefficients.

One way to attack this problem is to note that r^{n+2} is a monotonically increasing function of r and hence, using the mean value theorem in the Introduction,

$$R_n(z, \varphi) = \left[\frac{r^{n+2}(0)}{A_e^n} \int_0^{\bar{r}_n(\xi, \varphi)} \rho(r, z, \varphi)\, dr + \frac{A^{n+2}(z, \varphi)}{A_e^n} \int_{\bar{r}_n(z, \varphi)}^{A(z, \varphi)} \rho(r, z, \varphi)\, dr\right]$$

$$= \frac{A^{n+2}(z, \varphi)}{A_e^n} \int_{\bar{r}_n(z, \varphi)}^{A(z, \varphi)} \rho(r, z, \varphi)\, dr \tag{4.4}$$

This approach has the consequence of eliminating the factor $\frac{1}{n+3}$ which we are now accustomed to associate with the radial integral. This loss may, however, be compensated by the lower limit $\bar{r}_n(z, \varphi)$ on the last integral. If we transform the variable of integration by

$$\xi = \frac{r}{A_e} < 1 \quad , \quad \xi_{max}(z, \varphi) = \frac{A(z, \varphi)}{A_e} < 1 \quad , \quad \bar{\xi}_n(z, \varphi) = \frac{\bar{r}_n(z, \varphi)}{A_e} \tag{4.5}$$

then $R_n(z, \varphi)$ becomes [Eqs. (4.3) and (4.4)]

$$R_n(z, \varphi) = A_e^3 \int_0^{\xi_{max}(z, \varphi)} \rho(\xi, z, \varphi)\, \xi^{n+2}\, d\xi = A_e^3\, \xi_{max}^{n+2} \int_{\bar{\xi}_n(z, \varphi)}^{\xi_{max}(z, \varphi)} \rho(r, z, \varphi)\, d\varphi \tag{4.6}$$

We note that one factor, at least, of the integrand of the first integral decreases rapidly with increasing n over most of its interval of integration. On the other hand, ξ_{max}, the corresponding factor for the second integral is very close to unity and will decrease comparatively slowly for values of n of any immediate concern: for example, k must exceed 128 before $.9966^k$ drops below $\frac{1}{2}$ — .9966 is approximately the ratio of the polar to the equatorial radius of the Earth. One would thus expect $\bar{\xi}_n(z, \varphi)$ to approach $\xi_{max}(z, \varphi)$ fairly rapidly as n increases. To translate this expectation into a quantitative estimate of $\bar{\xi}_n(z, \varphi)$ is the difficult step in calculating bounds on $R_n(z, \varphi)$. In principle, $\bar{\xi}_n(z, \varphi)$ could be evaluated by equating the two integrals in Eq. (4.6) and solving for $\bar{\xi}_n(z, \varphi)$, but this is not feasible in practice.

We postpone, briefly, the discussion of this problem in order to investigate just what information on $\bar{\xi}_n(z, \varphi)$ might be required in order to place bounds on $R_n(z, \varphi)$. Using the second form of $R_n(z, \varphi)$ given in Eq. (4.6), we see that if

$$\Delta_n(z, \varphi) = \xi_{max}(z, \varphi) - \bar{\xi}_n(z, \varphi) \tag{4.7}$$

is "small," fairly tight bounds can be placed on $R_n(z, \varphi)$ as follows:

$$A_e^3\, \xi_{max}^{n+2}(z, \varphi)\, \Delta_n(z, \varphi)\, \bar{\rho}_{min}(z, \varphi) < R_n(z, \varphi) < A_e^3\, \xi_{max}^{n+2}(z, \varphi)\, \Delta_n(z, \varphi)\, \bar{\rho}_{max}(z, \varphi) \tag{4.8}$$

where $\bar{\rho}_{max}(z, \varphi)$ and $\bar{\rho}_{min}(z, \varphi)$ are maximum and minimum values of the density along the "ray" (z, φ) in the shell bounded by $\xi_{max}(z, \varphi)$ and $\bar{\xi}_n(z, \varphi)$. Now recall that what we need for Eq. (3.9) are the bounds $R_{ij, max}$ and $R_{ij, min}$ on $R_n(z, \varphi)$ for each pair of blocks $i, (j-1)$ and i, j. Introducing the bounds

$$\begin{aligned}
\Delta_{ij, min} &= GLB_{ij}\, \Delta_n(z, \varphi) \quad , & \Delta_{ij, max} &= LUB_{ij}\, \Delta_n(z, \varphi) \\
\xi_{ij, max} &= \frac{1}{A_e} LUB_{ij}\, A(z, \varphi) \quad , & \xi_{ij, min} &= \frac{1}{A_e} GLB_{ij}\, A(z, \varphi) \\
\bar{\rho}_{ij, min} &= GLB_{ij}\, \bar{\rho}_{min}(z, \varphi) \quad , & \bar{\rho}_{ij, max} &= LUB_{ij}\, \bar{\rho}_{max}(z, \varphi)
\end{aligned} \tag{4.9}$$

taken over block $i,(j-1)$ and its neighbor i,j (as done for R_n in Section III), we can write

$$A_e^3\,\xi_{ij,min}^{n+2}\,\Delta_{ij,min}\,\bar{\rho}_{ij,min} = R_{ij,min} < R_n(z,\varphi) < R_{ij,max} < A_e^3\,\xi_{ij,max}^{n+2}\,\Delta_{ij,max}\,\bar{\rho}_{ij,max} \qquad (4.10)$$

Estimates of $\xi_{ij,min}$ and $\xi_{ij,max}$ can probably be made with reasonable accuracy. If it turns out that $\Delta_{ij,max}$ is small, the estimation of $\bar{\rho}_{ij,max}$ and $\bar{\rho}_{ij,min}$ can be based on knowledge of the density function near the surface of the Earth where considerable direct observational data is available.

In any case, we see that $\Delta_n(z,\varphi)$ rather than $\bar{\xi}_n(z,\varphi)$ itself is the significant parameter, and "block" bounds, rather than point values, are what we need. As a first step, to get a feel for the order of magnitude of this parameter, we might consider a recent radial model, constructed by Wang,[7] for the density distribution of the Earth. In this model, the Earth is considered to be a sphere, the density depends only upon r, and a great deal of averaging is implicit in the construction. The basic idea would be to use Wang's model to approximate the two integrals of Eq. (4.6): the first for various values of n, and the second for various values of the lower limit ($\xi_{max} = 1$ for this model), and then by direct comparison, construct tables of values for $\bar{\xi}_n$ and $\Delta_n = 1 - \bar{\xi}_n$. Following this, one should carry out similar calculations based on more realistic models and/or construct density variations along selected "rays" (z,φ), for which a good supply of density data is available. Clearly, considerable analysis and numerical experimentation will be required to determine realistic estimates of $\Delta_{ij,max}$ and $\Delta_{ij,min}$.

It is possible that the theory of isostatic compensation can be used in the analysis. For if $\bar{\xi}_n(z,\varphi)$ turns out to be fairly close to, or below, the level of

compensation, it will follow that the second integral in Eq. (4.6) will be quite insensitive to z and φ, so that the block bounds $R_{ij,min}$ and $R_{ij,max}$ will be determined primarily by bounds for $[A(z, \varphi)]^n$.

V. "WORST" CASE ANALYSIS AND "BEST" RESULTS

In the estimation of bounds by rigorous procedures, such as those implemented by Cholshevnikov, inequality relationships are used and the question arises as to whether the final result "really" contains a $<$ or a $\leq$ condition. In some cases, the method of analysis clearly indicates that equality is not a possibility, as in Cholshevnikov's first and third theorems. If equality is not a possibility, then, clearly, the result is not "best": a sharper analysis should exist for which the bound can be tightened to the point where equality is a possibility. The realization of the condition of equality is a "worst case" because it, in effect, limits the further tightening of the bound.

One way to test whether or not a result is "best" is to try to construct the "worst case" consistent with the hypotheses of the analysis. This is the method employed by Cholshevnikov in showing that his Theorem II is a "best" result for $m = 0$. His "worst case" is a homogeneous hemisphere bounded by the $z = 0$ plane ("equatorial" plane) for which C_{n0} decreases as $1/n^3$. This very extreme density distribution is consistent with the hypothesis of bounded density with bounded variation of his theorem. Actually, the numerical factors for C_{n0} and the bounds of Theorem II do not quite match and neither does the exact dependence upon n; what Cholshevnikov is really claiming is a "qualitatively best" result.

Now, of course, we know that the Earth is not a homogeneous hemisphere; after all, the facts that $J_2 \sim 10^{-3}\mu$ and all other coefficients appear to be of order $\pm 10^{-6}\mu$ indicate that the Earth is nearly a homogeneous sphere. If constraints are imposed on the density function so that a "worst case" construction is limited to more or less realistic density models, it is highly unlikely that any of the theorems quoted in Section II would qualify as best results.

One interesting conjecture is confirmed both by Cholshevnikov's example and by the less extreme construction outlined below. The conjecture is that a worst case constructed to match a bound, or more accurately to approximate a bound, for some subset of coefficients, will most likely imply that many other coefficients are far below their theoretical bounds. In Cholshevnikov's example, all the tesseral coefficients C_{nm}, S_{nm} $(m > 0)$ vanish. A less extreme construction assumes realistic upper and lower bounds ρ_{min} and ρ_{max} for the density. Then a particular $C_{\overline{n}\overline{m}}$ (or $S_{\overline{n}\overline{m}}$) is maximized by assigning density ρ_{max} to all blocks for which the corresponding spherical harmonic is positive, and ρ_{min} to the rest. It is easy to show that this distribution implies that $S_{\overline{n}\overline{m}}$ (or $C_{\overline{n}\overline{m}}$) vanishes, and that (at least) all C_{nm} (or S_{nm}) with $m = 2k\overline{m}$ also vanish for k any integer > 0. Similar results can be obtained when an average value for ρ is imposed as an additional constraint.

"Worst case" constructions are useful in a number of ways although, as indicated above, some care must be exercised in the interpretation of the implications. Certainly such constructions are useful to obtain a "feel" for the problem in the early stages of analysis, and it is in this application that caution must be observed since such constructions are usually based on a very few "global" parameters. In later stages of the analysis when, perhaps, some results have been obtained using a limited data base, construction of a worst case can be revealing: the extent to which the worst case satisfies constraints imposed by data, which are not incorporated in the analysis, could be a measure of the practical value of the results. Finally, a worst case construction utilizing an extensive data base is a possible means of obtaining realistic bounds, and, in fact, the methods outlined in Sections III and IV might be regarded as algorithms for such constructions.

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